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Some Fifth and Sixth Order Iterative Methods for Solving Nonlinear Equations

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Abstract
In this paper, we derive multipoint iterative methods of fifth and sixth order for finding simple zeros of nonlinear equations. The methods are based on the composition of two steps – the first step consists of Jarratt fourth order method and the second is weighted Newton step to which correction term is applied. Per iteration each method requires two evaluations of the given function and two evaluations of its derivative. Numerical examples are presented to support that the methods thus obtained are competitive with Jarratt method. Moreover, it is shown that these methods are very useful in the applications requiring high precision in computations.
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I. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis and its applications. In fact, this problem is prototype for many nonlinear problems (see [15]). In this paper, we consider iterative methods to find simple roots of a nonlinear equation $f(x)=0$, where $f:I\subset \boldsymbol{R}\to \boldsymbol{R}$ is a scalar function on an open interval *I*.

Traub [14] has classified iterative methods into two categories viz. one-point iterative methods and multipoint iterative methods. Each of these is further divided into two subclasses, namely, one-point with and without memory and multipoint with and without memory. Investigation of one-point iterative methods with and without memory has demonstrated theoretical restrictions on the order and efficiency of this class of methods (see [14]). Neither of these restrictions need hold for multipoint iterative methods, that is, for the methods which sample function f and its derivatives at a number of values of independent variable. An additional feature of these techniques is that they may possess a number of free parameters which can be used to ensure, for example, that the convergence is of certain order and the sampling is done at felicitous points. The second condition is also a distinguishing characteristic of Gaussian quadrature integration formulae and Runge-Kutta methods for integrating ordinary differential equations.

These facts have led many researchers to investigate multipoint iterative methods. For example, Ostrowski developed third and fourth order methods [11, 14] requiring two function f and one derivative f' evaluations per step. The same information is required for King's family of fourth order methods [8]. Traub [14] suggested a third order method which requires one f and two f' evaluations. Jarratt [6] fourth order method requires the same number of information. King [7] introduced a fifth order scheme utilizing two evaluations of f and f'. Neta [9] developed a family of sixth order methods that requires the information of three f and one f'. Neta [10] also introduced a family of sixteenth order requiring four f and one f' per step. Jain [5] discussed a fifth order implicit method that uses the information of one f and three f'. Weerakoon and Fernando [16] developed a third order scheme requiring one f and two f' information. Ezquerro and Hernandez [2, 3] derived a family of third order methods requiring one f and two f' evaluations. Sharma and Goyal [12] suggested two derivative-free fourth order families of methods which require three f evaluations. Grau and Díaz-Barrero [4] developed an improved sixth order Ostrowski method that requires the information of three f and one f'. Recently, Sharma and Guha [13] have derived a family of such methods which requires the same information. All of these methods are classified as multipoint methods without memory in which, except [12], a Newton or weighted Newton step is followed by a faster Newton-like step. However, in method [12] Steffensen step is followed by a faster Steffensen-like step.

In particular, Jarratt fourth order multipoint method [6] is given by

$$x_{n+1}=x_n-\frac{1}{2}\frac{f(x_n)}{f'(x_n)}-\frac{f(x_n)}{3f'(y_n)-f'(x_n)}, \qquad n=0,1,2,3\ldots\ldots\ldots \tag{1}$$

where $y_n=x_n-\frac{2}{3}\frac{f(x_n)}{f'(x_n)}$. The method requires one function f and two derivatives f' evaluations per step. This formula is particularly useful for the problems in which evaluation of f' is cheaper compared with f, such as the function defined by an integral.

In this paper, we develop a one-parameter (i.e. $a\in \boldsymbol{R}-\{-1\}$) family of multipoint methods based on the composition of two steps, Jarratt step (1) which is a predictor-type and weighted Newton step which is a corrector-type. As a consequence, the order of convergence is improved from four for Jarratt method to six for the new methods. Per iteration the new methods require two evaluations of the function f and two of its derivative f'. Further we find that the choice $a=-1$ of the parameter results in fifth order method. These formulae are tested and performance is compared with Jarratt method.

II. Development of the methods

Our aim is to develop a scheme that improves the order of convergence of Jarratt method (1) at the cost of an additional evaluation of function. Thus we begin with the following iterative scheme

$$\left.\begin{aligned} z_n&=x_n-\frac{1}{2}\frac{f(x_n)}{f'(x_n)}-\frac{f(x_n)}{3f'(y_n)-f'(x_n)}\\ x_{n+1}&=z_n-\frac{f'(x_n)+af'(y_n)}{bf'(x_n)+cf'(y_n)}\frac{f(z_n)}{f'(x_n)}\end{aligned}\right\}, \tag{2}$$

where $a, b,$ c are parameters and y_n is as defined in (1).

This scheme consists of a Jarratt step to get z_n from x_n, followed by a weighted Newton step to calculate x_{n+1} from the point z_n. The parameters a, b and c used in (2) can be determined from the following theorem:

Theorem. *Let $f:I\subset \boldsymbol{R}\to\boldsymbol{R}$ denote a real valued function defined on I, where I is a neighborhood of a simple root α of $f(x)$. Assume that $f(x)$ is sufficiently smooth in the interval I. Then, the iterative scheme (2) defines a one-parameter family of sixth order convergence if $b=-\frac{1}{2}(3a+1),\ c=\frac{1}{2}(5a+3),$ provided $a\neq-1$. If $a=-1$, the scheme is of fifth order.*

Proof. Using Taylor expansion of $f(x_n)$ about α and taking into account that $f(\alpha)=0,\ f'(\alpha)\neq0,$ we have

$$f(x_n)=f'(\alpha)[e_n+A_2e_n^2+A_3e_n^3+A_4e_n^4+o(e_n^5)], \tag{3}$$

where $e_n=x_n-\alpha$ and $\mathrm{A_k}=(1/k!)\,f^{(k)}(\alpha)/f'(\alpha),\ k=2,3,\ldots\ldots\ldots$

Furthermore, we have $$f'(x_n)=f'(\alpha)[1+2A_2e_n+3A_3e_n^2+4A_4e_n^3+o(e_n^4)], \tag{4}$$

and $$\frac{f(x_n)}{f'(x_n)}=e_n-A_2e_n^2+2(A_2^2-A_3)e_n^3+(7A_2A_3-4A_2^3-3A_4)e_n^4+o(e_n^5). \tag{5}$$

Substituting (5) in $y_n=x_n-(2/3)\,f(x_n)/f'(x_n)$ yields

$$y_n-\alpha=\frac{1}{3}e_n+\frac{2}{3}A_2e_n^2-\frac{4}{3}(A_2^2-A_3)e_n^3-\frac{2}{3}(7A_2A_3-4A_2^3-3A_4)e_n^4+o(e_n^5)\,. \tag{6}$$

Expanding $f'(y_n)$ about α and using (6), we have

$$f'(y_n)=f'(\alpha)\left[1+\frac{2}{3}A_2e_n+\frac{1}{3}(4A_2^2+A_3)e_n^2-4(\frac{2}{3}A_2^3-A_2A_3-\frac{1}{27}A_4)e_n^3+o(e_n^4)\right], \tag{7}$$

From (4) and (7), we get

$$3f'(y_n)-f'(x_n)=2f'(\alpha)\left[1+(2A_2^2-A_3)e_n^2-2(2A_2^3-3A_2A_3+\frac{8}{9}A_4)e_n^3+o(e_n^4)\right]. \tag{8}$$

Dividing (3) by (8) and simplifying, we obtain

$$\frac{f(x_n)}{3f'(y_n)-f'(x_n)}=\frac{1}{2}\left[e_n+A_2e_n^2+2(-A_2^2+A_3)e_n^3+(2A_2^3-5A_2A_3+\frac{25}{9}A_4)e_n^4+o(e_n^5)\right]. \tag{9}$$

Substituting (5) and (9) in the first step of (2), we have

$$z_n=x_n-\frac{1}{2}\frac{f(x_n)}{f'(x_n)}-\frac{f(x_n)}{3f'(y_n)-f'(x_n)}=\alpha+(A_2^3-A_2A_3+\frac{1}{9}A_4)e_n^4+o(e_n^5), \tag{10}$$

that is $$z_n-\alpha=(A_2^3-A_2A_3+\frac{1}{9}A_4)e_n^4+o(e_n^5). \tag{11}$$

From (4) and (7), we obtain

$$f'(x_n)+af'(y_n)=f'(\alpha)\Big[1+a+2(1+\frac{1}{3}a)A_2e_n+\left(\frac{4}{3}aA_2^2+3(1+\frac{1}{9}a)A_3\right)e_n^2 \\ -4\left(\frac{2}{3}aA_2^3-aA_2A_3-(1+\frac{1}{27}a)A_4\right)e_n^3+o(e_n^4)\Big], \tag{12}$$

and $$bf'(x_n)+cf'(y_n)=f'(\alpha)\Big[b+c+2(b+\frac{1}{3}c)A_2e_n+\left(\frac{4}{3}bA_2^2+3(b+\frac{1}{9}c)A_3\right)e_n^2 \\ -4\left(\frac{2}{3}cA_2^3-cA_2A_3-(b+\frac{1}{27}c)A_4\right)e_n^3+o(e_n^4)\Big]. \tag{13}$$

Upon dividing (12) by (13) and simplifying, we get

$$\frac{f'(x_n)+af'(y_n)}{bf'(x_n)+cf'(y_n)}=\frac{1}{b+c}\Big[1+a+\frac{4(c-ab)}{b+c}\Big\{\frac{1}{3}A_2e_n-\frac{1}{b+c}\Big((b+\frac{5}{9}c)A_2^2 \\ -\frac{2}{3}(b+c)A_3\Big)e_n^2\Big\}+o(e_n^3)\Big]. \tag{14}$$

Now, the Taylor expansion of $f(z_n)$ about α

$$f(z_n)=f'(\alpha)[(z_n-\alpha)+o\{(z_n-\alpha)^2\}]. \tag{15}$$

From (4), (11) and (15) we find that

$$\frac{f(z_n)}{f'(x_n)}=[1-2A_2e_n+(4A_2^2-3A_3)e_n^2](z_n-\alpha)+o(e_n^7). \tag{16}$$

Multiplication of (14) and (16) yields

$$\frac{f'(x_n)+af'(y_n)}{bf'(x_n)+cf'(y_n)}\frac{f(z_n)}{f'(x_n)}=\frac{1}{b+c}\Big[1+a-\frac{2}{b+c}\Big(b(1+\frac{5}{3}a)+c(a+\frac{1}{3})\Big)A_2e_n \\ +\frac{1}{b+c}\Big\{\frac{4}{b+c}\Big(b^2(1+\frac{8}{3}a)+c^2(a-\frac{2}{9})+\frac{1}{3}bc(1+\frac{29}{3}a)\Big)A_2^2 \\ -3\Big(b(1+\frac{17}{9}a)+c(a+\frac{1}{9})\Big)A_3\Big\}e_n^2\Big](z_n-\alpha)+o(e_n^7). \tag{17}$$

Thus making use of (11) and (17) in the second step of (2), we obtain

$$e_{n+1}=\Big[1-\frac{1+a}{b+c}+\frac{2}{(b+c)^2}\Big(b(1+\frac{5}{3}a)+c(a+\frac{1}{3})\Big)A_2e_n \\ -\frac{1}{(b+c)^2}\Big\{\frac{4}{b+c}\Big(b^2(1+\frac{8}{3}a)+c^2(a-\frac{2}{9})+\frac{1}{3}bc(1+\frac{29}{3}a)\Big)A_2^2 \\ -3\Big(b(1+\frac{17}{9}a)+c(a+\frac{1}{9})\Big)A_3\Big\}e_n^2\Big]\Big[A_2^3-A_2A_3+\frac{1}{9}A_4\Big]e_n^4+o(e_n^7). \tag{18}$$

In order to find a correction term for the second step of (2) such that the proposed scheme may yield sixth order method, the coefficients of e_n^4 and e_n^5 must vanish. Hence from (18), we derive the following conditions on the parameters a, b and c

$$-a+b+c=1 \quad \text{and} \quad 2b(1+a)+(1+3a)(b+c)=0 \tag{19}$$

Since (19) represents a set of two equations in three unknown parameters, we may solve for any two of the parameters in terms of the remaining one. Solving for b and c in terms of a, we obtain

$$b=-\frac{1}{2}(3a+1) \quad \text{and} \quad c=\frac{1}{2}(5a+3). \tag{20}$$

With these values and from (18), the error equation turns out to be

$$e_{n+1}=\left[-\frac{2a-6}{3a+3}A_2^2-A_3\right]\left[A_2^3-A_2A_3+\frac{1}{9}A_4\right]e_n^6+o(e_n^7). \tag{21}$$

Thus equation (21) establishes the maximum order of convergence equal to six for the iteration scheme (2), provided $a\neq-1$. However, if $a=-1$, then from (19) we have $b=-c$ and consequently the second step of (2) reduces to

$$x_{n+1}=z_n-\frac{1}{b}\frac{f(z_n)}{f'(x_n)}, \tag{22}$$

thereby using (11) and (16) in (22), we get the error equation

$$e_{n+1}=\left[\left(1-\frac{1}{b}\right)+\frac{2}{b}A_2e_n\right]\left[A_2^3-A_2A_3+\frac{1}{9}A_4\right]e_n^4+o(e_n^6). \tag{23}$$

So it follows that for the scheme (22) to be of fifth order, $b=1$. In this case the error is

$$e_{n+1}=2A_2\left[A_2^3-A_2A_3+\frac{1}{9}A_4\right]e_n^5+o(e_n^6). \tag{24}$$

Thus we establish that for the parametric values $a=-b=c=-1$, the scheme (2) is of fifth order. This completes the proof of the theorem. □

Hence the proposed scheme (2) with the parameter $a\in\boldsymbol{R}$ is given by

$$\left.\begin{aligned} z_n&=x_n-\frac{1}{2}\frac{f(x_n)}{f'(x_n)}-\frac{f(x_n)}{3f'(y_n)-f'(x_n)}\\ x_{n+1}&=z_n-2\frac{af'(y_n)+f'(x_n)}{(5a+3)f'(y_n)-(3a+1)f'(x_n)}\frac{f(z_n)}{f'(x_n)}\end{aligned}\right\}, \tag{25}$$

where $y_n=x_n-\dfrac{2}{3}\dfrac{f(x_n)}{f'(x_n)}$.

This scheme is of fifth order if $a=-1$, otherwise it is of sixth order. Thus, we have derived a method of fifth order and a one-parameter family of sixth order methods by an additional evaluation of function at the point iterated by Jarratt method. Such processes are valuable since, as Traub [14, Ch. 11] has shown, they may be readily generalized to deal with the problem of solving systems of equations.

In order to obtain an assessment of the efficiency of our methods we shall make use of Traub's efficiency index ([14], Appendix C), according to which efficiency of an iterative method is given by $E=\log p/C$, where p is the order of the method and C is cost per iterative step of computing the function and derivative values needed to form the iterative formula. If we let the cost of evaluating f be c_f and that of f' be $c_{f'}$ then for our fifth and sixth order formulae the efficiencies are $E=\log 5/2(c_f+c_{f'})$ and $E=\log 6/2(c_f+c_{f'})$, respectively. For Jarratt method, we have similarly $E=\log 4/(c_f+2c_{f'})$. Comparing the E values we find that sixth order method will be a better choice than Jarratt if $c_f<0.82\,c_{f'}$. That means if the cost of calculating f is less than 82 percent of that necessary to calculate f', then sixth order method is more efficient. Similar comparison shows that if $c_f < 0.38 c_{f'}$, then fifth order method is superior to Jarratt method.

III. Numerical examples

Here we shall apply our methods to the following nonlinear equations:

$$f_1 = x^4 - 6x^3 + 11x^2 - 6x = 0, \quad x_0 = 5.0$$
$$f_2 = x^2 - (1-x)^5 = 0, \quad x_0 = 1.0$$
$$f_3 = x\,e^{x^2} - \sin^2 x + 3\cos x + 5 = 0, \quad x_0 = -1.0$$
$$f_4 = \sin x\, e^{-x} + \ln(x+1) = 0, \quad x_0 = -0.5$$
$$f_5 = \tan^{-1} x - x + 1 = 0, \quad x_0 = 1.0$$
$$f_6 = e^{x^2+7x-30} - 1 = 0, \quad x_0 = 3.$$
$$f_7 = \int_0^x \frac{\sin xt}{t}\,dt - \frac{1}{2} = 0, \quad x_0 = 0.2$$

where x_0 is the initial approximation chosen.

The performance is compared with Jarratt method which is now designated as M_1. Also designating fifth and sixth order methods as M_2 and M_3, respectively. The parameter a that appears in the algorithm M_3 is chosen $a = 1$. All calculations are performed in double precision arithmetic. We accept an approximate solution rather than the exact root, depending on the computer precision $(\in)$. The stopping criteria used for computer program: (i) $|x_{i+1} - x_i| < \in$, (ii) $|f(x_{i+1})| < \in$, and so, when the stopping criterion is satisfied, x_{i+1} is taken as the computed root α. For numerical illustrations in this section, we use fixed stopping criterion $\in = 0.5\times10^{-17}$.

Table 1 shows the calculated root and the number of iterations necessary to reach the root up to the desired accuracy by each method. In table 2, we obtain costs of the present methods in comparison with the classical predecessor M_1 for each problem. The cost of an iterative method is defined by $c_t = nc_i$, where n is the number of iterations, c_i is the cost per iteration and c_t is the total cost. We assume that the cost of evaluating one iterative step is primarily due to the cost of evaluating $f^{(j)}$, $j \geq 0$ and that the cost of combining $f^{(j)}$ to form iteration method is negligible. In particular, we have assumed the cost of evaluating each $f^{(j)}$ is 1 in all the considered problems.

Table 1. Results of the problems.

Problem	Root	Iterations (n)		
		M_1	M_2	M_3
f_1	3.0000000000000000	4	4	3
f_2	0.3459548158482421	4	3	3
f_3	-1.2076478271309190	3	3	2
f_4	0.0000000000000000	3	3	2
f_5	2.1322677252728850	3	3	2
f_6	3.0000000000000000	6	5	3
f_7	0.7121746839007776	4	4	3

Table 2 Cost of methods compared to Jarrat method

Problem	M_1	M_2	M_3
f_1	1	1.33	1
f_2	1	1	1
f_3	1	1.33	0.89
f_4	1	1.33	0.89
f_5	1	1.33	0.89
f_6	1	1.11	0.67
f_7	1	1.33	1

IV. Concluding remarks

In this paper, we have suggested multipoint methods of order fifth and sixth using an additional evaluation of function at the point iterated by Jarratt method of order four for solving equations. Thus, one requires two evaluations of the function f and two of its first-derivative f' per full step in the application of each method. A reasonably close starting value is necessary for the methods to converge. This condition, however, practically applies to all iterative methods for solving equations.

The numerical results overwhelmingly support that the new methods improve the order of convergence of Jarratt method. Comparison of the costs of the methods shows that sixth order method is cheaper in the problems where f is easier to evaluate than f', such as f defined by transcendental function e.g. problems $f_3 - f_6$, whereas fifth order method is expensive in general. Jarratt method, however, is better if f is defined by either polynomial function or integral function, for example, in the case of problems f_1, f_2 and f_7. Finally, we conclude the paper with the remark that these higher order methods may be very useful in the applications requiring multiprecision in their computations because such methods yield a clear reduction in the iterations.

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